

# TRANSVERSE INSTABILITY FOR PERIODIC WAVES OF KP-I AND SCHRÖDINGER EQUATIONS

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**ABSTRACT.** We consider the quadratic and cubic KP - I and NLS models in 1+2 dimensions with periodic boundary conditions. We show that the spatially periodic travelling waves (with period  $K$ ) in the form  $u(t, x, y) = \varphi(x - ct)$  are spectrally and linearly unstable, when the perturbations are taken to be with the same period. This strong instability implies other instabilities considered recently - for example with respect to perturbations with periods  $nK, n = 2, 3, \dots$  or bounded perturbations.

## 1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

The existence and stability properties of special solutions of nonlinear differential equations is an important question both from theoretical and practical point of view. Many equations describing wave motion typically feature traveling wave solutions. The problem of the orbital stability of solitary waves for nonlinear dispersive equations goes back to the works of Benjamin [8] and Bona [9]. Another approach is to linearize the equation around the solitary wave and look for linear stability based on the spectrum of the linear solution operator. Extending the ODE ideas to partial differential equations has introduced a number of new issues. In infinite dimensions, the relation between the linearization and the full nonlinear equations is far more complicated. Another nontrivial issue arises at the linear level, since all of the known proofs for the existence of invariant manifolds are based upon the use of the solution group (or semigroup) generated by the linearization. However, in any actual problem, the information available will, at best, be of the spectrum of the infinitesimal generator, that is, the linearized equation and not its solution operator. Relating the spectrum of the infinitesimal generator to that of the group is a spectral mapping problem that is often non-trivial. All of these three problems - spectral stability, linear stability and nonlinear stability, have been extensively studied for solitary wave solutions.

While the existence and stability of such solutions on the whole space case has been well-studied, the questions about existence and stability of spatially periodic traveling waves have not received much attention until recently. One of the first results on stability of periodic solutions of the Korteweg-de Vries(KdV) equation was obtained by McKean [25]. Based on the integrability of the KdV equation the stability of all periodic finite-genus solutions has been established. Recently Angulo, Bona and Scialom [4] investigated the orbital stability of cnoidal waves for the KdV equation with respect to perturbations of the same period. The linear stability/instability of some of these solutions with respect to different types of perturbations has been developed in the last couple of years, see for example [18], [11] and [10]. Other new explicit formulae

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for periodic traveling waves of dnoidal type together with their stability have been obtained in [3, 5, 14, 15, 16].

An interesting aspect of the theory is when one considers the one-dimensional waves as solutions in two-dimensional models. One generally refers to this as the question for *transverse stability* of such waves. The transverse stability of traveling waves is associated with a class of perturbations traveling transversely to the direction of the basic traveling wave.

The problem of transverse stability/instability of solitary waves goes back to a work by Kadomtsev and Petviashvili [23] for KdV solitary waves. It turns out that solitary waves are transverse stable in the case of KP-I and transverse unstable in the case of KP-II.

Recently, Rousset and Tzvetkov, [28, 29] provided general criterium for transverse instability for traveling waves of Hamiltonian partial differential equations, which was then applied to various examples. Johnson and Zumbrun [22] investigated the stability *periodic* traveling waves of the generalized KdV equation to two dimensional perturbations, which are nonperiodic (bounded) in the generalized KP equation and have long wavelength in transverse direction. By analyzing high and low frequency limits of the appropriate periodic Evans function they derived an instability criterion for the transverse instability. This criteria is then applied to the KdV and modified KdV equations. The authors proved that the periodic traveling waves of the KdV equation are unstable to long wavelength transverse perturbations and that cnoidal, and dnoidal traveling waves for modified KdV equation are transverse unstable to long wavelength perturbations in KP-II and KP-I respectively. Haragus [17] considered the transverse spectral stability of *small periodic* traveling wave solutions of the KdV equation with respect to perturbations in KP-I and KP-II which are either periodic in the direction of perturbation or nonperiodic (localized or bounded) and have long wavelength in the transverse direction.

In this paper, we prove transverse instability of certain periodic solutions of the Kadomtsev-Petviashvili-I equation and the nonlinear Schrödinger equation. More precisely, we consider periodic traveling waves of the KdV and mKdV equation, which in turn also solve the KP-I equation, while our second example concerns spatially periodic standing waves of the non-linear Schrödinger equation (NLS). Before we continue with the specifics of our results, we outline the general scheme and we give some definitions.

In this paper we only deal with the stability information provided by the linearized equation<sup>1</sup>. Suppose that the linearized equation is in the form of an evolution equation

$$(1) \quad v_t = \mathcal{A}v.$$

We use the following definition of spectral and linear stability

**Definition 1.** Assume that  $\mathcal{A} = \mathcal{A}(\varphi)$  generates a  $C_0$  semigroup on a Banach space  $X$ . We say that the solution  $\varphi$  with linearized problem (1) is *spectrally stable*, if  $\sigma(\mathcal{A}) \subset \{\lambda : \Re \lambda \leq 0\}$ .

We say that the the solution  $\varphi$  with linearized problem (1) is *linearly stable*, if the growth bound for the semigroup  $e^{t\mathcal{A}}$  is non-positive. Equivalently, we require that every solution of (1) with  $v(0) \in X$  has the property

$$\lim_{t \rightarrow \infty} e^{-\delta t} \|v(t, \cdot)\| = 0$$

for every  $\delta > 0$ .

**Remarks:** We recall that by the spectral mapping theorem for point spectrum  $\sigma_{p.p.}(e^{t\mathcal{A}}) \setminus \{0\} = e^{t\sigma_{p.p.}(\mathcal{A})}$ . There is however only the inclusion  $\sigma_{ess}(e^{t\mathcal{A}}) \setminus \{0\} \supseteq e^{t\sigma_{ess}(\mathcal{A})}$ , which is the reason that one cannot, in general (and in the absence of the so-called spectral mapping theorem), deduce

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<sup>1</sup>i.e. we will not consider the full non-linear equation satisfied by  $v$ , which would of course amount to non-linear stability/instability results.

linear stability from spectral stability. In fact, due to the spectral inclusions above, linear stability implies spectral stability, but in general the converse is false.

However, in the cases considered in this paper the spectrum consists entirely of eigenvalues and the two notions are equivalent (since there is a spectral mapping theorem for eigenvalues, as indicated above). Thus, we will concentrate on the spectral stability from now on.

**1.1. KP - I equation.** Consider the spatially periodic KP - I equation

$$(2) \quad \begin{cases} (u_t + \partial_{xxx}u + \partial_x(f(u)))_x - \partial_{yy}u = 0, & (t, x, y) \in \mathbf{R}_+^1 \times [0, K_1] \times [0, K_2] \\ u(t, x + K_1, y) = u(t, x, y); u(t, x, y + K_2) = u(t, x, y) \end{cases}$$

where  $f$  is smooth function<sup>2</sup>. It is known that solutions exists, at least locally, when the data is in the product Sobolev spaces  $f \in H^{3,3}([0, K_1] \times [0, K_2])$ , see for example [21].

In this paper, we will be interested in the stability properties of a class of special solutions, namely the periodic traveling waves solution of the modified KdV equation. That is, we look for solutions in the form  $v(t, x) = \varphi(x - ct)$ ,  $\varphi(x + K_1) = \varphi(x)$ , so that

$$v_t + \partial_{xxx}v + \partial_x(f(v)) = 0, \quad x \in [0, K_1].$$

Clearly then  $u(t, x, y) := \varphi(x - ct)$  is a solution of the KP - I equation (2). We construct these solutions  $\varphi_c$  explicitly in Section 2 below. Periodic travelling-wave solution are determined from Newton's equation which we will write below in the form  $\varphi'^2 = U(\varphi)$ . Therefore by using the well-known properties of the phase portrait of Newton's equation in the  $(\varphi, \varphi')$ -plane, one can establish that under fairly general conditions, that there exists a family of periodic solutions  $\varphi(y) = \varphi(c, \varphi_0; y)$  and  $\varphi_0 = \min \varphi$ . Moreover, if  $T = T(c, \varphi_0)$  (in particular, their period turns out to depend on the speed parameter  $c$  and an elliptic modulus  $\kappa$ ) is the minimal (sometimes called *fundamental*) period of  $\varphi$ , then  $\varphi$  has exactly one local minimum and one local maximum in  $[0, T)$ . Therefore  $\varphi'$  has just two zeroes in each semi-open interval of length  $T$ . By Floquet theory, this means that  $\varphi'$  is either the second or the third eigenfunction of the periodic eigenvalue problem.

In order to explain the stability/instability results, we need to linearize the equation (2) about the periodic traveling wave solution. Namely, write an ansatz in the form  $u(t, x, y) = \varphi(x - ct) + v(t, x - ct, y)$ , which we plug in (2). After ignoring all nonlinear in  $v$  terms, we arrive at the following linear equation for  $v$

$$(3) \quad (v_t + v_{xxx} - cv + (f'(\varphi)v)_x)_x - \partial_{yy}v = 0.$$

If the variable  $v$  has the mean-zero property in  $x$  (i.e.  $\int_0^{K_1} v(t, x, y) dx = 0$ ), then one may invert the operator  $\partial_x$  (by defining  $(\partial_x^{-1}f)(x) := \int_0^x f(y) dy$ ) and thus recast (3) in the evolution equation form

$$(4) \quad v_t = \partial_x(-\partial_x^2 + c - f'(\varphi))v + \partial_x^{-1}\partial_{yy}v$$

The question for stability/instability of traveling wave solutions of the KP - I equation has attracted a lot of attention in the last few years (see [22], [17]).

As we have indicated above, we restrict our attention to spectral considerations for the generator. In order to establish instability, we seek solutions in the form

$$v(t, x, y) = e^{\sigma t} e^{iky} V(x),$$

where  $\sigma \in \mathbb{C}$ ,  $k \in \mathbb{R}$  and  $V(x)$  is periodic function with same period as the periodic traveling wave solution  $\varphi(x)$ . Clearly, such solutions will be also periodic in the  $y$  variable, with period

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<sup>2</sup>We only consider the cases  $f(u) = u^2, \pm u^3$ , but other choices certainly make sense mathematically.

$K_2 = 2\pi/k$ . Thus, if we manage to show existence of such  $V = V(\sigma, k)$  with some  $\sigma > 0$ , we will have shown transeverse spectral instability of the traveling wave solution  $\varphi(x)$ .

We further specialize  $V$  in the form  $V = \partial_x U$ . Plugging in (4) yields the equation

$$-\sigma \partial_x U = (-\partial_x(-\partial_{xx} + c - f'(\varphi))\partial_x + k^2)U.$$

This eigenvalue problem is therefore in the form

$$(5) \quad \sigma A(k)U = L(k)U,$$

with

$$A(k) = -\partial_x, \quad L(k) = -\partial_x(-\partial_{xx} + c - f'(\varphi))\partial_x + k^2.$$

where  $L(k), A(k)$  are operators which depend on the real parameter  $k$  on some Hilbert space  $H$ .

**1.2. The Nonlinear Schrödinger Equation.** Another object of investigation will be the spacially periodic solutions of the Nonlinear Schrödinger Equation (NLS).

$$(6) \quad \begin{cases} iu_t - (u_{xx} + u_{yy}) - f(|u|^2)u = 0, (t, x, y) \in \mathbf{R}_+^1 \times [0, K_1] \times [0, K_2] \\ u(t, x + K_1, y) = u(t, x, y); \quad u(t, x, y + K_2) = u(t, x, y). \end{cases}$$

where  $f$  is a smooth function. Looking for standing waves in the form  $u(t, x) = e^{-i\omega t}\varphi(x)$  results in the ordinary differential equation

$$(7) \quad \omega\varphi - \varphi'' - f(\varphi^2)\varphi = 0.$$

We now derive the linearized equation for small perturbation of the wave  $e^{-i\omega t}\varphi$ . Write the ansatz  $u = e^{-i\omega t}(\varphi + v(t, x, y))$ . For the nonlinear term, we have

$$f(|u|^2) = f(|\varphi + v|^2) = f(\varphi^2 + 2\varphi\Re v + |v|^2) = f(\varphi^2) + 2f'(\varphi^2)\varphi\Re v + O(v^2).$$

We get, after disregarding  $O(v^2)$  terms and taking into account (7),

$$iv_t + \omega v - (v_{xx} + v_{yy}) - f(\varphi^2)v - 2f'(\varphi^2)\varphi^2\Re v = 0.$$

We are looking for unstable solutions in the form  $v(t, x, y) = e^{\sigma t} \cos(ky)V(x)$ , where  $V$  is a complex-valued function. We obtain

$$i\sigma V + \omega V - V'' + k^2 V - f(\varphi^2)V - 2f'(\varphi^2)\varphi^2\Re V = 0$$

Let  $V = v_1 + iv_2$ , where  $v_1, v_2$  are real-valued functions. This gives the following system for  $v_1, v_2$

$$\begin{aligned} \sigma v_1 - v_2'' + \omega v_2 + k^2 v_2 - f(\varphi^2)v_2 &= 0 \\ -\sigma v_2 - v_1'' + \omega v_1 + k^2 v_1 - f(\varphi^2)v_1 - 2f'(\varphi^2)\varphi^2 v_1 &= 0. \end{aligned}$$

Denote

$$\begin{aligned} \mathcal{L}_+ &= -\partial_x^2 + \omega - f(\varphi^2), \\ \mathcal{L}_- &= -\partial_x^2 + \omega - f(\varphi^2) - 2f'(\varphi^2)\varphi^2. \end{aligned}$$

This allows us to write the linearized problem as follows

$$(8) \quad \sigma \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{L}_+ + k^2 \\ -(\mathcal{L}_- + k^2) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

Let  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ . Note that  $J^* = J^{-1} = -J$ . In terms of  $z_1, z_2$ , we have the equation

$$\sigma J \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -J \begin{pmatrix} \mathcal{L}_- + k^2 & 0 \\ 0 & \mathcal{L}_+ + k^2 \end{pmatrix} J \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Thus, we have managed to recast the problem in the form (5), this time with

$$(9) \quad A(k) = \sigma J; \quad L(k) = -J \begin{pmatrix} \mathcal{L}_- + k^2 & 0 \\ 0 & \mathcal{L}_+ + k^2 \end{pmatrix} J = -J \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} J + k^2 Id$$

**Remark:** We would like to give the important case  $f(z) = \sqrt{z}$  some more consideration, due to the fact that the function  $\sqrt{z}$  fails to be differentiable at zero. Nevertheless, we still have

$$\sqrt{|\varphi + v|} = \varphi + \Re v + o(v),$$

and we still obtain the formula

$$\begin{aligned} \mathcal{L}_+ &= -\partial_x^2 + \omega - \varphi, \\ \mathcal{L}_- &= -\partial_x^2 + \omega - 2\varphi, \end{aligned}$$

as we would, if we were to use the derivative of the function  $f(z)$  in the generic definition of  $\mathcal{L}_\pm$  above. The difference of course is in the fact that the remainder term is only  $o(v)$  instead of  $O(v^2)$ , but this of course is irrelevant for the linear theory that we develop here.

**1.3. Main Results.** Our first result concerns the transverse instability of the cnoidal solutions of the KP - I equation.

**Theorem 1.** *(transverse instability for cnoidal solutions of KP - I)*

Considering the KP - I equation (i.e. (2) with  $f(u) = \frac{u^2}{2}$ ). It supports the cnoidal solutions given by (23) below. Then, there exists a period  $K_2$  depending on the particular cnoidal solution, so that the cnoidal waves are spectrally and linearly unstable for all values of the parameters  $\kappa \in (0, 1)$  and  $T$  given by (24).

Next, we state our main result regarding transverse instability of the dnoidal solutions of the modified KP - I equation.

**Theorem 2.** *(transverse instability for dnoidal solutions of modified KP - I)*

Consider the modified KP - I equation, that is (2) with  $f(u) = u^3$ . Then, there exists a period  $K_2$  depending on the particular dnoidal solution, so that the dnoidal solutions described by (32) below are spectrally and linearly unstable for all values of the parameters  $\kappa \in (0, 1)$  and the corresponding  $T$ .

Finally, we have the following result, which shows transverse instability for standing waves of the quadratic and cubic NLS. That is, we shall be considering (6) with  $f(z) = \sqrt{z}$  and  $f(z) = z$ .

**Theorem 3.** *(transverse instability for standing wave solutions of NLS)*

The quadratic (focussing) Schrödinger equation<sup>3</sup> (6) admits cnoidal solutions in the form (41). There exists  $K_2$ , depending on the specific solution, so that these solutions are spectrally and linearly unstable for all values of the parameter  $\kappa \in (0, 1)$ .

The cubic (focussing) Schrödinger equation<sup>4</sup> (6) supports dnoidal solutions in the form (46). There exists  $K_2$ , depending on the specific solution, so that these solutions are spectrally and linearly unstable for all values of the parameter  $\kappa \in (0, 1)$ .

#### Remarks:

- As a consequence of the three theorems above, one may deduce spectral instability, when the perturbations are taken to be periodic (with period equal to integer times the period of the wave) or bounded functions.

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<sup>3</sup>i.e.  $f(z) = \sqrt{z}$

<sup>4</sup>i.e.  $f(z) = z$

- Our method for showing transverse instability fails for periodic snoidal waves of the defocussing modified KP - I equation, see Chapter 5. Beyond the technical issues, which prevents the relevant inequality (37) from being satisfied, it would be interesting to further investigate the transverse stability/instability of these interesting waves.

**1.4. General instability criteria.** In our proofs, we use the following sufficient condition for instability.

**Theorem 4.** *Assume that the operator  $L(k)$  satisfies<sup>5</sup>*

- (1) *there exists  $k_0 > 0$ , so that  $\dim \text{Ker}[L(k_0)] = 1$ , say  $\text{Ker}[L(k_0)] = \text{span}\{\varphi\}$ .*
- (2)  *$L'(k_0)\varphi \neq 0$ .*

*Then, the equation (5) has a solution  $U$  for some  $k$ , sufficiently close to  $k_0$  and for some sufficiently small  $\sigma > 0$ . In fact, there exists a continuous scalar function  $k(\sigma) : k(0) = k_0$  and a continuous  $H$ -valued function  $U(\sigma) : U(0) = \varphi$ , so that*

$$\sigma A(k(\sigma))U(\sigma) = L(k(\sigma))U(\sigma),$$

*for all  $0 < \sigma \ll 1$ .*

**Note:** This is a variant of a theorem used by Groves-Haragus and Sun, [13]. The interested reader should also explore the simple exposition in [30], where several examples about transverse instability on the whole space are worked out in detail using the same techniques.

*Proof.* We quickly indicate the main ideas of the proof.

Let  $U = \varphi + V$ , with

$$V \in \varphi^\perp = \{V \in H, (V, \varphi) = 0\}.$$

Consider the equation  $G(V, k, \sigma) = 0$ , with  $\sigma > 0$  and

$$G(V, k, \sigma) = L(k)\varphi + L(k)V - \sigma A(k)\varphi - \sigma A(k)V.$$

We have

$$\langle D_{V,k}(0, k_0, 0), [\omega, \mu] \rangle = \mu L'(k_0)\varphi + L(k_0)\omega$$

and  $D_{V,k}(0, k_0, 0)$  is a bijection from  $\varphi^\perp \times \mathbb{R}$  to  $H$ . Thus from the implicit function theorem follows that for  $\sigma$  in a neighborhood of zero there exists  $k(\sigma)$  and  $V(\sigma)$  such that  $G(V(\sigma), k(\sigma), \sigma) = 0$ .  $\square$

Clearly, in view of Theorem 4 and the spectral problem (5), we will have proved Theorem 1 and Theorem 2, provided we can verify the conditions (1), (2) of Theorem 4 for the operator

$$L(k) = -\partial_x \mathcal{L} \partial_x + k^2 = -\partial_x (-\partial_{xx} + c - f'(\varphi)) \partial_x + k^2.$$

Similarly, for Theorem 3, due to the representation (9), it suffices to verify conditions (1), (2) of Theorem 4 for the operator

$$L(k) = \mathcal{J}^{-1} \mathcal{L} \mathcal{J} + k^2 = \mathcal{J}^{-1} \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} \mathcal{J} + k^2.$$

This clearly necessitates a somewhat detailed study of the spectral picture for the operators  $\mathcal{L}, \mathcal{L}_\pm$ . Luckily, after one constructs the traveling/standing waves for our models in terms of elliptic functions, we will be able to obtain some information about the spectra of  $\mathcal{L}$  and  $\mathcal{L}_\pm$ , which will allow us to check condition (1) in Theorem 4.

The paper is organized as follows. In Section 2, we construct the eigenfunctions. In Section 3, we describe the structure of the first few eigenvalues, together with the associated eigenfunctions

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<sup>5</sup>Hereafter, we use the notation  $L'(k) := \frac{d}{dk} L(k)$ .

for  $\mathcal{L}$  and  $\mathcal{L}_\pm$ . In section 4, we give the proof of Theorem 1 by verifying conditions (1), (2). This requires some spectral theory, together with the specific spectral information for  $\mathcal{L}, \mathcal{L}_\pm$ , obtained in Section 2. In section 5, we show that an identical approach for the defocusing modified KP - I equation *fails* to give transverse instability. Thus, an interesting question is left open, namely - Are the snoidal solutions to this problem transverse unstable?

## 2. CONSTRUCTION OF PERIODIC TRAVELING WAVES

We are looking for a traveling-wave solution for the equation

$$(10) \quad u_t + (f(u))_x + u_{xxx} = 0$$

of the form  $u(x, t) = \phi(x - ct)$ . We assume that  $\phi$  is smooth and bounded in  $\mathbb{R}$ . The following two cases appear:

- (i)  $\phi' \neq 0$  in  $\mathbb{R}$  and  $\phi_- < \phi < \phi_+$  (corresponding to kink-wave solution);
- (ii)  $\phi'(\xi) = 0$  for some  $\xi \in \mathbb{R}$ . Denote  $\phi_0 = \phi(\xi)$ ,  $\phi_2 = \phi''(\xi)$ .

Below we will deal with the second case. Replacing in (10) we get

$$(11) \quad -c\phi' + (f(\phi))' + \phi''' = 0.$$

Integrating (10) twice, one obtains

$$(12) \quad -c\phi + f(\phi) + \phi'' = a$$

$$(13) \quad \frac{\phi'^2}{2} = b + a\phi + \frac{c}{2}\phi^2 - F(\phi), \quad F(\phi) = \int_0^\phi f(s)ds$$

with some constants  $a, b$ . In case (ii), one has respectively

$$\begin{aligned} a &= f(\phi_0) - c\phi_0 + \phi_2, \\ b &= F(\phi_0) - \frac{1}{2}c\phi_0^2 - a\phi_0 = F(\phi_0) - \frac{1}{2}c\phi_0^2 - (-c\phi_0 + f(\phi_0) + \phi_2)\phi_0. \end{aligned}$$

Next we are going to look for periodic travelling-wave solutions  $\phi$ . Consider in the plane  $(X, Y) = (\phi, \phi')$  the Hamiltonian system

$$(14) \quad \begin{aligned} \dot{X} &= Y = H_Y, \\ \dot{Y} &= -f(X) + cX + a = -H_X, \end{aligned}$$

with a Hamiltonian function

$$H(X, Y) = \frac{Y^2}{2} + F(X) - \frac{c}{2}X^2 - aX.$$

Then (13) becomes  $H(\phi, \phi') = b$  and the curve  $s \rightarrow (\phi(s - s_0), \phi'(s - s_0))$  determined by (13) lies on the energy level  $H = b$  of the Hamiltonian  $H(X, Y)$ . Within the analytical class, system (14) has periodic solutions if and only if it has a center. Each center is surrounded by a continuous band of periodic trajectories (called *period annulus*) which terminates at a certain separatrix contour on the Poincaré sphere. The critical points of center type of (14) are given by the critical points on  $Y = 0$  having a negative Hessian. These are the points  $(X_0, 0)$  where:

$$(15) \quad a + cX_0 - f(X_0) = 0, \quad c - \phi'(X_0) < 0.$$

(For simplicity, we will not consider here the case of a degenerate center when the Hessian becomes zero.)

The above considerations lead us to the following statement.

**Proposition 1.** *Let  $a$  and  $c$  be constants such that conditions (15) are satisfied for some  $X_0 \in \mathbb{R}$ . Then there is an open interval  $\Delta$  containing  $X_0$  such that:*

- (i) *For any  $\phi_0 \in \Delta$ ,  $\phi_0 < X_0$ , the solution of (10) satisfying*

$$\phi(\xi) = \phi_0, \quad \phi'(\xi) = 0, \quad \phi''(\xi) = a + c\phi_0 - f(\phi_0),$$

*is periodic.*

- (ii) *If  $\phi_1 \in \Delta$ ,  $\phi_1 > X_0$  is the nearest to  $X_0$  solution of  $H(X, 0) = H(\phi_0, 0)$ , then  $\phi_0 \leq \phi \leq \phi_1$ .*

- (iii) *If  $T$  is the minimal period of  $\phi$ , then in each interval  $[s, s + T)$ , the function  $\phi$  has just one minimum and one maximum ( $\phi_0$  and  $\phi_1$ , respectively) and it is strictly monotone elsewhere.*

Denote

$$U(s) = 2b + 2as + cs^2 - 2F(s) = 2F(\phi_0) - c\phi_0^2 - 2a\phi_0 + 2as + cs^2 - 2F(s).$$

Then for  $\phi_0 \leq \phi \leq \phi_1$  one can rewrite (13) as  $\phi'(\sigma) = \sqrt{U(\phi(\sigma))}$ . Integrating the equation along the interval  $[\xi, s] \subset [\xi, \xi + T/2]$  yields an implicit formula for the value of  $\phi(s)$ :

$$(16) \quad \int_{\phi_0}^{\phi(s)} \frac{d\sigma}{\sqrt{U(\sigma)}} = s - \xi, \quad s \in [\xi, \xi + T/2].$$

For  $s \in [\xi + T/2, \xi + T]$  one has  $\varphi(s) = \varphi(T + 2\xi - s)$ . We recall that the period function  $T$  of a Hamiltonian flow generated by  $H_0 \equiv \frac{1}{2}Y^2 - \frac{1}{2}U(X) = 0$  is determined from

$$(17) \quad T = \int_0^T dt = \oint_{H_0=0} \frac{dX}{Y} = 2 \int_{\varphi_0}^{\varphi_1} \frac{dX}{\sqrt{U(X)}}.$$

This is in fact the derivative (with respect to the energy level) of the area surrounded by the periodic trajectory through the point  $(\phi_0, 0)$  in the  $(X, Y) = (\phi, \phi')$ -plane.

Consider the continuous family of periodical travelling-wave solutions  $\{u = \phi(x - ct)\}$  of (10) and (12) going through the points  $(\phi, \phi') = (\phi_0, 0)$  where  $\phi_0 \in \Delta^-$ . For any  $\phi_0 \in \Delta^-$ , denote by  $T = T(\phi_0)$  the corresponding period. One can see (e.g. by using formula (16) above) that the period function  $\phi_0 \rightarrow T(\phi_0)$  is smooth. To check this, it suffices to perform a change of the variable

$$(18) \quad X = \frac{\phi_1 - \phi_0}{2}s + \frac{\phi_1 + \phi_0}{2}$$

in the integral (16) and use that

$$(19) \quad U(\varphi_0) = U(\varphi_1) = 0.$$

Conversely, taking  $c, a$  to satisfy the conditions of Proposition 1 and fixing  $T$  in a proper interval, one can determine  $\phi_0$  and  $\phi_1$  as smooth functions of  $c, a$  so that the periodic solution  $\phi$  given by (16) will have a period  $T$ . The condition for this is the monotonicity of the period (for more details see).

### 3. SPECTRAL PROPERTIES OF THE OPERATORS $\mathcal{L}$ AND $\mathcal{L}_\pm$

We first construct the spectral representation of the KdV equation



**3.1. The operator  $\mathcal{L}$  for KdV.** Consider the Korteweg-de Vries equation

$$(20) \quad u_t + uu_x + u_{xxx} = 0,$$

which is a particular case of (10) with  $f(u) = \frac{u^2}{2}$ . In this subsection we are interested of the spectral properties of the operator  $\mathcal{L}$  defined by the

$$(21) \quad \mathcal{L} = -\partial_x^2 + c - \phi.$$

Let us first mention that (14) reduces now to

$$X_0 = c + \sqrt{c^2 + 2}, \quad \Delta = \left( c - \sqrt{c^2 + 2}, c + 2\sqrt{c^2 + 2} \right)$$

By the definition of  $a, b$  and  $U(s)$  one obtains

$$\begin{aligned} U(s) &\equiv \frac{1}{3}(\phi_0 - s)[s^2 + (\phi_0 - 3c)s - (2\phi_0^2 + 3c\phi_0 - 6\phi_2)] \\ &= \frac{1}{3}(s - \phi_0)(\phi_1 - s)(s + \phi_1 + \phi_0 - 3c). \end{aligned}$$

We note that the last equality is a consequence of Proposition 1, which implies that  $U(\phi_1) = U(\phi_0) = 0$ . To obtain an explicit formula for the travelling wave  $\phi_c$ , we substitute  $\sigma = \phi_0 + (\phi_1 - \phi_0)z^2$ ,  $z > 0$  in order to express the above integral as an elliptic integral of the first kind in a Legendre form. One obtains

$$\int_0^{Z(s)} \frac{dz}{\sqrt{(1-z^2)(\kappa'^2 - k^2 z^2)}} = \alpha(s - \xi),$$

where

$$(22) \quad Z(s) = \sqrt{\frac{\phi_c(s) - \phi_0}{\phi_1 - \phi_0}}, \quad k^2 = \frac{\phi_1 - \phi_0}{\phi_0 + 2\phi_1 - 3c}, \quad \kappa^2 + \kappa'^2 = 1, \quad \alpha = \sqrt{\frac{\phi_0 + 2\phi_1 - 3c}{12}}.$$

Thus we get the expression

$$(23) \quad \phi_c(s) = \phi_0 + (\phi_1 - \phi_0)cn^2(\alpha(s - \xi); k).$$

To calculate the period of  $\phi_c$ , we use (3.7) and the same procedure as above. In this way we get

$$(24) \quad T = 2 \int_{\phi_0}^{\phi_1} \frac{d\sigma}{\sqrt{U(\sigma)}} = \frac{2}{\alpha} \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} = \frac{2K(k)}{\alpha}.$$

We return to the operator  $\mathcal{L}$  defined by (21), where  $\phi_c$  is determined by (23). Consider the spectral problem

$$(25) \quad \begin{aligned} \mathcal{L}\psi &= \lambda\psi, \\ \psi(0) &= \psi(T), \quad \psi'(0) = \psi'(T). \end{aligned}$$

We will denote the operator just defined again by  $\mathcal{L}$ . It is a self-adjoint operator acting on  $L_{per}^2[0, T]$  with  $D(\mathcal{L}) = H^2([0, T])$ . From the Floquet theory applied to (25) it follows that its spectrum is purely discrete,

$$(26) \quad \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$$

where  $\lambda_0$  is always a simple eigenvalue. If  $\psi_n(x)$  is the eigenfunction corresponding to  $\lambda_n$ , then

$$(27) \quad \begin{aligned} \psi_0 &\text{ has no zeroes in } [0, T]; \\ \psi_{2n+1}, \psi_{2n+2} &\text{ have each just } 2n+2 \text{ zeroes in } [0, T]. \end{aligned}$$

**Proposition 2.** *The linear operator  $\mathcal{L}$  defined by (25) has the following spectral properties:*

- (i) *The first three eigenvalues of  $\mathcal{L}$  are simple.*
- (ii) *The second eigenvalue of  $\mathcal{L}$  is  $\lambda_1 = 0$ .*
- (iii) *Remainder of the spectrum is consisted by a discrete set eigenvalues.*

**Proof.** By (11),  $\mathcal{L}\phi'_c = 0$ , hence  $\psi = \phi'_c$  is an eigenfunction corresponding to zero eigenvalue. By Proposition 1 (iii)  $\phi'$  has just two zeroes in  $[0, T)$  and therefore by (27) either  $0 = \lambda_1 < \lambda_2$  or  $\lambda_1 < \lambda_2 = 0$  or  $\lambda_1 = \lambda_2 = 0$ . We are going to verify that only the first possibility  $0 = \lambda_1 < \lambda_2$  can occur. From the definition of  $k$  and  $\alpha$  one obtains that

$$\phi_0 + 2\phi_1 - 3c = 12\alpha^2, \quad \phi_1 - \phi_0 = 12k^2\alpha^2.$$

Then using (23) we get

$$\begin{aligned} \mathcal{L} &= -\partial_x^2 + c - \phi_0 - (\phi_1 - \phi_0)cn^2(\alpha x; k) \\ &= -\partial_x^2 + c - \phi_1 + (\phi_1 - \phi_0)sn^2(\alpha x; k) \\ &= -\partial_x^2 - \alpha^2[4k^2 + 4 - 12k^2sn^2(\alpha x; k)] \\ &= \alpha^2[-\partial_y^2 - 4k^2 - 4 + 12k^2sn^2(y; k)] \equiv \alpha^2\Lambda \end{aligned}$$

where  $y = \alpha x$ . The operator  $\Lambda$  is related to Hill's equation with Lamé potential

$$\Lambda w = -\frac{d^2}{dy^2}w + [12k^2sn^2(y; k) - 4k^2 - 4]w = 0$$

and its spectral properties in the interval  $[0, 2K(k)]$  are well known [4, ?, ?]. The first three (simple) eigenvalues and corresponding periodic eigenfunctions of  $\Lambda$  are

$$\begin{aligned} \mu_0 &= k^2 - 2 - 2\sqrt{1 - k^2 + 4k^4} < 0, \\ \psi_0(y) &= dn(y; k)[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4})sn^2(y; k)] > 0, \\ \mu_1 &= 0, \\ \psi_1(y) &= dn(y; k)sn(y; k)cn(y; k) = \frac{1}{2}(d/dy)sn^2(y; k), \\ \mu_2 &= k^2 - 2 + 2\sqrt{1 - k^2 + 4k^4} > 0, \\ \psi_2(y) &= dn(y; k)[1 - (1 + 2k^2 + \sqrt{1 - k^2 + 4k^4})sn^2(y; k)]. \end{aligned}$$

As the eigenvalues of  $\mathcal{L}$  and  $\Lambda$  are related by  $\lambda_n = \alpha^2\mu_n$  we conclude that the first three eigenvalues of (25) are simple and moreover  $\lambda_0 < 0$ ,  $\lambda_1 = 0$ ,  $\lambda_2 > 0$ . The corresponding eigenfunctions are  $\psi_0(\alpha x)$ ,  $\psi_1(\alpha x) = \text{const.}\phi'_c(x)$  and  $\psi_2(\alpha x)$ .  $\square$

**3.2. The operator  $\mathcal{L}_{mKV}$ .** Consider the modified Korteweg-de Vries equation

$$(28) \quad u_t + 3u^2u_x + u_{xxx} = 0.$$

Traveling wave solutions in this case satisfy the equation

$$(29) \quad -c\phi' + 3\phi^2\phi' + \phi''' = 0.$$

Integrating yields

$$(30) \quad \phi'' = a + c\phi - \phi^3.$$

We consider the "symmetric" case  $a = 0$  only. Integrating once again, we get

$$(31) \quad \phi'^2 = b + c\phi^2 - \frac{\phi^4}{2}.$$

Hence the periodic solutions are given by the periodic trajectories  $H(\phi, \phi') = b$  of the Hamiltonian vector field  $dH = 0$  where

$$H(x, y) = y^2 + \frac{x^4}{4} - c\frac{x^2}{2}.$$

Then there are two possibilities

1.1) (*outer case*): for any  $b > 0$  the orbit defined by  $H(\phi, \phi') = b$  is periodic and oscillates around the eight-shaped loop  $H(\phi, \phi') = 0$  through the saddle at the origin.

1.2) (*left and right cases*): for any  $b \in (-\frac{1}{2}c^2, 0)$  there are two periodic orbits defined by  $H(\phi, \phi') = b$  (the left and right ones). These are located inside the eight-shaped loop and oscillate around the centers at  $(\pm\sqrt{c}, 0)$ , respectively.

We will consider the left and right cases of Duffing oscillator.

In the left and the right cases, let us denote by  $\phi_1 > \phi_0 > 0$  the positive roots of the quartic equation  $\frac{\phi^4}{2} - a\phi^2 - b = 0$ . Then, up to a translation, we obtain the respective explicit formulas

$$(32) \quad \phi(z) = \mp \phi_1 dn(\alpha z; k), \quad k^2 = \frac{\phi_1^2 - \phi_0^2}{\phi_1^2} = \frac{2\phi_1^2 - 2c}{\phi_1^2}, \quad \alpha = \frac{\phi_1}{\sqrt{2}}, \quad T = \frac{2K(k)}{\alpha}.$$

Now

$$(33) \quad \mathcal{L} = -\partial_x^2 + c - 3\phi^2.$$

We use (32) to rewrite the operator  $\mathcal{L}$  in an appropriate form. From the expression for  $\phi(x)$  from (32) and the relations between the elliptic functions  $sn(x)$ ,  $cn(x)$  and  $dn(x)$ , we obtain

$$\mathcal{L} = \alpha^2[-\partial_y^2 + 6k^2 sn^2(y) - 4 - k^2]$$

where  $y = \alpha x$ .

It is well-known that the first five eigenvalues of  $\Lambda = -\partial_y^2 + 6k^2 sn^2(y, k)$ , with periodic boundary conditions on  $[0, 4K(k)]$ , where  $K(k)$  is the complete elliptic integral of the first kind, are simple. These eigenvalues, with their corresponding eigenfunctions are as follows

$$\begin{aligned} \nu_0 &= 2 + 2k^2 - 2\sqrt{1 - k^2 + k^4}, & \psi_0(y) &= 1 - (1 + k^2 - \sqrt{1 - k^2 + k^4})sn^2(y, k), \\ \nu_1 &= 1 + k^2, & \psi_1(y) &= cn(y, k)dn(y, k) = sn'(y, k), \\ \nu_2 &= 1 + 4k^2, & \psi_2(y) &= sn(y, k)dn(y, k) = -cn'(y, k), \\ \nu_3 &= 4 + k^2, & \psi_3(y) &= sn(y, k)cn(y, k) = -k^{-2}dn'(y, k), \\ \nu_4 &= 2 + 2k^2 + 2\sqrt{1 - k^2 + k^4}, & \psi_4(y) &= 1 - (1 + k^2 + \sqrt{1 - k^2 + k^4})sn^2(y, k). \end{aligned}$$

It follows that the first three eigenvalues of the operator  $\mathcal{L}$ , equipped with periodic boundary condition on  $[0, 2K(k)]$  (that is, in the case of left and right family), are simple and  $\lambda_0 = \alpha^2(\nu_0 - \nu_3) < 0$ ,  $\lambda_1 = \alpha^2(\nu_3 - \nu_3) = 0$ ,  $\lambda_2 = \alpha^2(\nu_4 - \nu_3) > 0$ . The corresponding eigenfunctions are  $\chi_0 = \psi_0(\alpha x)$ ,  $\chi_1 = \phi'(x)$ ,  $\chi_2 = \psi_4(\alpha x)$ .

Thus, we have proved the following

**Proposition 3.** *The linear operator  $\mathcal{L}$  defined by (33) has the following spectral properties:*

- (i) *The first three eigenvalues of  $\mathcal{L}$  are simple.*
- (ii) *The second eigenvalue of  $\mathcal{L}$  is  $\lambda_1 = 0$ , which is simple.*
- (iii) *The rest of the spectrum consists of a discrete set of eigenvalues, which are strictly positive.*

## 4. PROOF OF THEOREM 1

We first consider the cases of the KdV and the modified KdV equations.

**4.1. The KdV and modified KdV equations.** We need to check the assumptions of Theorem 4 for the operator  $L(k) = -\partial_x \mathcal{L} \partial_x + k^2$ , where  $\mathcal{L}$  is either the operator associated to the KdV equation, constructed in Section 3.1 or the operator associated to the mKdV equation, constructed in Section 3.2.

Clearly,  $L(0) = -\partial_x \mathcal{L} \partial_x$  is bounded from below self-adjoint operator, so that its spectrum consists of eigenvalues with finite multiplicities

$$\sigma(L(0)) = \lambda_0(L(0)) \leq \lambda_1(L(0)) \leq \dots$$

Thus, one may apply the Courant principle for the first eigenvalue. We have

$$\lambda_1(L(0)) = \sup_{z \neq 0} \inf_{u \perp z} \frac{\langle L(0)u, u \rangle}{\|u\|^2}.$$

and as a consequence the infimum in  $u$  may be taken only on functions with mean value zero. Taking  $z = \psi'_0$  and the identity  $\langle L(0)u, u \rangle = \langle -\partial_x \mathcal{L} \partial_x u, u \rangle = \langle \mathcal{L}u', u' \rangle$  allows us to write

$$\lambda_1(L(0)) \geq \inf_{u \perp \psi'_0} \frac{\langle \mathcal{L}u', u' \rangle}{\|u\|^2} = \inf_{u' \perp \psi_0} \frac{\langle \mathcal{L}u', u' \rangle}{\|u\|^2}$$

since  $\langle u', \psi_0 \rangle = -\langle u, \psi'_0 \rangle = 0$ . Now, observe that since in both  $\mathcal{L}_{KdV}$  and  $\mathcal{L}_{mKdV}$  we have that there is only a single and simple negative eigenvalue, it follows that  $\mathcal{L}|_{\{\psi_0\}^\perp} \geq 0$ , i.e.  $\langle \mathcal{L}v, v \rangle \geq 0$ , whenever  $v \perp \psi_0$ . In particular, if  $u' \perp \psi_0$ ,

$$\frac{\langle \mathcal{L}u', u' \rangle}{\|u\|^2} \geq 0.$$

Thus,

$$\lambda_1(L(0)) \geq \inf_{u' \perp \psi_0} \frac{\langle \mathcal{L}u', u' \rangle}{\|u\|^2} \geq 0.$$

Thus  $\lambda_1(L(0)) \geq 0$ . On the other hand, we have that 0 is an eigenvalue for  $L(0)$ , because  $L(0)\phi_c = -\partial_x \mathcal{L} \partial_x [\phi_c] = -\partial_x \mathcal{L} \phi'_c = 0$ .

We will now show that there is a negative eigenvalue for  $L_{KdV}(0)$  and  $L_{mKdV}(0)$ . We claim that this will be enough for the proof of Theorem 1.

Indeed, if we succeed in showing  $\lambda_0(L(0)) < 0$ , and since we have established  $0 \in \sigma(L(0))$  and  $\lambda_1(L(0)) \geq 0$ , it follows that  $\lambda_1(L(0)) = 0$ . In particular  $\lambda_0(L(0))$  is a simple eigenvalue, hence verifying the first hypothesis of Theorem 4 with  $k_0^2 := -\lambda_0(L(0))$ . Moreover,  $L'(k_0) = 2k_0 Id$  and hence, the second condition of Theorem 4 is trivially satisfied as well.

Thus, it suffices to show

$$(34) \quad \lambda_0(L(0)) = \inf_{u: \|u\|=1} \langle \mathcal{L}u', u' \rangle < 0$$

Here we present a sufficient condition for this to happen. Namely, we construct  $u' := t_0 \psi_0 - t_2 \psi_2$  for some coefficient  $t_0, t_2$  to be found momentarily. To that end, we shall first need

$$(35) \quad t_0 \int_0^T \psi_0(y) dy - t_2 \int_0^T \psi_2(y) dy = 0$$

to ensure that such a periodic function  $u$  exists<sup>6</sup>. Since both  $\int_0^T \psi_0(y)dy \neq 0$ ,  $\int_0^T \psi_2(y)dy \neq 0$ , we conclude that we may select  $t_0, t_2 \neq 0$  and

$$(36) \quad \frac{t_0}{t_2} = \frac{\int_0^T \psi_2(y)dy}{\int_0^T \psi_0(y)dy}.$$

Next, we compute (using (36))

$$\begin{aligned} \langle \mathcal{L}u', u' \rangle &= \langle \mathcal{L}(t_0\psi_0 - t_2\psi_2), (t_0\psi_0 - t_2\psi_2) \rangle = t_0^2 \|\psi_0\|_{L^2}^2 \lambda_0(\mathcal{L}) + t_2^2 \|\psi_2\|_{L^2}^2 \lambda_2(\mathcal{L}) = \\ &= t_2^2 \left( \int_0^T \psi_2(y)dy \right)^2 \left( \frac{\|\psi_0\|^2 \lambda_0(\mathcal{L})}{\left( \int_0^T \psi_0(y)dy \right)^2} + \frac{\|\psi_2\|^2 \lambda_2(\mathcal{L})}{\left( \int_0^T \psi_2(y)dy \right)^2} \right). \end{aligned}$$

Thus, it remains to check that under the conditions in Theorem 4, the following inequality holds true

$$(37) \quad \frac{\|\psi_0\|^2 \lambda_0(\mathcal{L})}{\left( \int_0^T \psi_0(y)dy \right)^2} + \frac{\|\psi_2\|^2 \lambda_2(\mathcal{L})}{\left( \int_0^T \psi_2(y)dy \right)^2} < 0.$$

Thus, we have reduced the proof of Theorem 1 and Theorem 2 to checking (37) for  $\mathcal{L}_{KdV}$  and  $\mathcal{L}_{mKdV}$  respectively.

4.1.1. *Proof of (37) for  $\mathcal{L}_{KdV}$ .* In the case of Korteweg-de Vries equation using (22) and identities

$$\begin{aligned} sn^2(x) &= \frac{1}{\kappa^2}(1 - dn^2(x)) \\ \int_0^K dn(x)dx &= \frac{\pi}{2} \\ \int_0^K dn^3(x)dx &= \frac{\pi(2-\kappa^2)}{4} \\ \int_0^K dn^2(x)sn^2(x)dx &= \frac{(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)}{3\kappa^2} \\ \int_0^K dn^2(x)sn^4(x)dx &= \frac{(8\kappa^4-3\kappa^2-2)E(\kappa)+2(1+\kappa^2-2\kappa^4)K(\kappa)}{15\kappa^4} \end{aligned}$$

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<sup>6</sup>in which case, we simply define the non-trivially zero function  $u(x) := \int_0^x (t_0\psi_0(y) - t_2\psi_2(y))dy$ , which in view of (35) is defined up to a multiplicative constant.

we get<sup>7</sup>

$$\begin{aligned}
\int_0^T \psi_0(\alpha x) dx &= \frac{\pi}{\alpha} \left[ \frac{2-\kappa^2}{2\kappa^2} (1 + 2\kappa^2 - \sqrt{1-\kappa^2+4\kappa^4}) + \frac{\sqrt{1-\kappa^2+4\kappa^4}-1-\kappa^2}{\kappa^2} \right] \\
\int_0^T \psi_0^2(\alpha x) dx &= \frac{2}{\alpha} \left( E(\kappa) + \frac{2(-1-2\kappa^2+\sqrt{1-\kappa^2+4\kappa^4})((-1+2\kappa^2)E(\kappa)-(-1+\kappa^2)K(\kappa))}{3\kappa^2} \right. \\
&\quad \left. + \frac{(-1-2\kappa^2+\sqrt{1-\kappa^2+4\kappa^4})^2((-2-3\kappa^2+8\kappa^4)E(\kappa)+2(1+\kappa^2-2\kappa^4)K(\kappa))}{15\kappa^4} \right) \\
\int_0^T \psi_2(\alpha x) dx &= \frac{\pi}{\alpha} \left[ \frac{2-\kappa^2}{2\kappa^2} (1 + 2\kappa^2 + \sqrt{1-\kappa^2+4\kappa^4}) - \frac{1+\kappa^2+\sqrt{1-\kappa^2+4\kappa^4}}{\kappa^2} \right] \\
\int_0^T \psi_2^2(\alpha x) dx &= \frac{2}{\alpha} \left( E(\kappa) + \frac{2(1+2\kappa^2+\sqrt{1-\kappa^2+4\kappa^4})((1-2\kappa^2)E(\kappa)+(-1+\kappa^2)K(\kappa))}{3\kappa^2} \right. \\
&\quad \left. + \frac{(1+2\kappa^2+\sqrt{1-\kappa^2+4\kappa^4})^2((-2-3\kappa^2+8\kappa^4)E(\kappa)+2(1+\kappa^2-2\kappa^4)K(\kappa))}{15\kappa^4} \right) \\
\lambda_0(\mathcal{L}) &= \alpha^2(k^2 - 2 - 2\sqrt{1-k^2+4k^4}) \\
\lambda_2(\mathcal{L}) &= \alpha^2(k^2 - 2 + 2\sqrt{1-k^2+4k^4}),
\end{aligned}$$

where  $\alpha$  is given by (22). Thus, we have an explicit formula to work with in order to show (37).

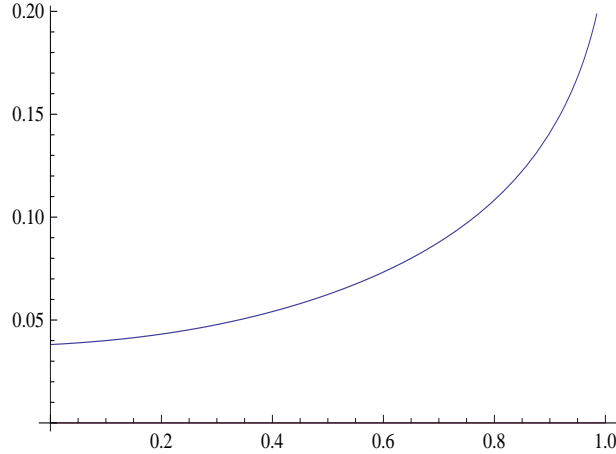


FIGURE 1. This is a graph of the function  $h(\kappa) = \frac{\int_0^{2K(\kappa)} \psi_2(x) dx}{\sqrt{\lambda_2} \|\psi_2\|} - \frac{\int_0^{2K(\kappa)} \psi_0(x) dx}{\sqrt{\lambda_0} \|\psi_0\|}$ . Note that positivity of  $h$  is equivalent to the validity of (37).

From the graph in Figure 1, it is clear that the inequality (37) holds for all values of the parameter  $\kappa$ .

<sup>7</sup>In the derivation of the formulas below, we have used the symbolic integration feature of the *Mathematica* software.

4.1.2. *Proof of (37) for  $\mathcal{L}_{mKdV}$ .* In the case of Modified Korteweg-de Vries equation using (32) and identities

$$sn^2(x) = \frac{1}{\kappa^2}(1 - dn^2(x))$$

$$\int_0^K dn^2(x)dx = E(\kappa)$$

$$\int_0^K sn^4(x)dx = \frac{1}{3\kappa^4} [(2 + \kappa^2)K(\kappa) - 2(1 + \kappa^2)E(\kappa)]$$

we get that

$$\int_0^T \psi_0(\alpha x)dx = \frac{2}{\alpha\kappa^2}(\sqrt{1 - \kappa^2 + \kappa^4} - 1)K(\kappa) + (1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4})E(\kappa)$$

$$\begin{aligned} \int_0^T \psi_0^2(\alpha x)dx &= \frac{2}{\alpha} \left( K(\kappa) - 2(1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4}) \frac{K(\kappa) - E(\kappa)}{\kappa^2} \right. \\ &\quad \left. + (1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4})^2 \frac{(2 + \kappa^2)K(\kappa) - 2(1 + \kappa^2)E(\kappa)}{3\kappa^4} \right) \end{aligned}$$

$$\int_0^T \psi_2(\alpha x)dx = \frac{2}{\alpha\kappa^2}(\sqrt{1 - \kappa^2 + \kappa^4} + 1 + \kappa^2)E(\kappa) - (1 + \sqrt{1 - \kappa^2 + \kappa^4})K(\kappa)$$

$$\begin{aligned} \int_0^T \psi_2^2(\alpha x)dx &= \frac{2}{\alpha} \left( K(\kappa) - 2(1 + \kappa^2 + \sqrt{1 - \kappa^2 + \kappa^4}) \frac{K(\kappa) - E(\kappa)}{\kappa^2} \right. \\ &\quad \left. + (1 + \kappa^2 + \sqrt{1 - \kappa^2 + \kappa^4})^2 \frac{(2 + \kappa^2)K(\kappa) - 2(1 + \kappa^2)E(\kappa)}{3\kappa^4} \right) \end{aligned}$$

$$\lambda_0 = \alpha^2(k^2 - 2 - 2\sqrt{1 - k^2 + k^4})$$

$$\lambda_2 = \alpha^2(k^2 - 2 + 2\sqrt{1 - k^2 + k^4}),$$

where  $\alpha$  is given by (32). Now the inequality 37 is satisfied for all  $\kappa \in (0, 1)$ . Again, the graph below shows that the inequality (37) is satisfied for all values of the parameter  $\kappa$ .

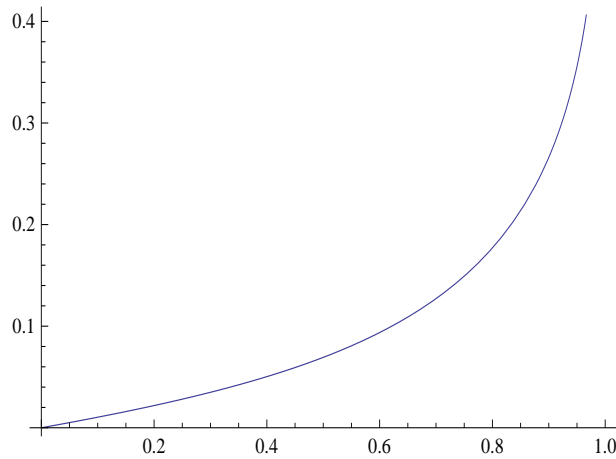


FIGURE 2. This is a graph of the function  $h(\kappa) = \frac{\int_0^{2K(\kappa)} \psi_2(x)dx}{\sqrt{\lambda_2}\|\psi_2\|} - \frac{\int_0^{2K(\kappa)} \psi_0(x)dx}{\sqrt{|\lambda_0|}\|\psi_0\|}$ . Note that positivity of  $h$  is equivalent to the validity of (37).

**4.2. The nonlinear Schrödinger equation.** In this section we will construct the periodic traveling wave solution for the quadratic and cubic nonlinear Schrödinger equations and investigate the spectral problems for corresponding operators. The results can be found in [16], but for convenience we will present here. We show that the matrix operator

$$\begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix}$$

has a single simple negative eigenvalue. The same will be true for the similar operator  $-J \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} J = J^{-1} \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} J$ . Thus, according to the instability criterium in Theorem 4 and the representation (9), this implies that we can select a  $k$  so that the operator  $L(k)$  satisfies (1) and (2), whence we will have shown spectral instability.

**4.2.1. Quadratic Schrödinger equation.** Consider the quadratic equation

$$(38) \quad iu_t + u_{xx} + |u|u = 0$$

for a complex-valued function  $u$ .

For  $\varphi$  one obtains the equation (7), which is

$$\varphi'' - \omega\varphi + \varphi|\varphi| = 0.$$

Therefore,

$$(39) \quad \varphi'^2 - \omega\varphi^2 + \frac{2}{3}\varphi^2|\varphi| = c$$

and  $\varphi$  is periodic provided that the level set  $H(x, y) = c$  of the Hamiltonian system  $dH = 0$ ,

$$H(x, y) = y^2 - \omega x^2 + \frac{2}{3}x^2|x|,$$

contains a periodic trajectory (an oval). The level set  $H(x, y) = c$  contains two periodic trajectories if  $\omega > 0$ ,  $c \in (-\frac{1}{3}\omega^3, 0)$  and a unique periodic trajectory if  $\omega \in \mathbb{R}$ ,  $c > 0$ . Under these conditions, equation (39) becomes  $H(\varphi, \varphi') = c$  and its solution  $\varphi$  is periodic of period  $T = T(\omega, c)$ .

Below, we consider the case  $c < 0$ . Then either  $\varphi < 0$  (the left case) or  $\varphi > 0$  (the right case). To express  $\varphi$  through elliptic functions, we denote by  $\varphi_0 > \varphi_1 > 0$  the positive solutions of  $\frac{2}{3}\rho^3 - \omega\rho^2 - c = 0$ . Then  $\varphi_1 \leq |\varphi| \leq \varphi_0$  and one can rewrite (39) as

$$(40) \quad \varphi'^2 = \frac{2}{3}(|\varphi| - \varphi_1)(\varphi_0 - |\varphi|)(|\varphi| + \varphi_0 + \varphi_1 - \frac{3}{2}\omega).$$

Therefore  $2\varphi_0 + \varphi_1 > \varphi_0 + 2\varphi_1 > \frac{3}{2}\omega$ . Introducing a new variable  $s \in (0, 1)$  via  $|\varphi| = \varphi_1 + (\varphi_0 - \varphi_1)s^2$ , we transform (40) into

$$s'^2 = \alpha^2(1 - s^2)(k'^2 + k^2s^2)$$

where  $\alpha, k, k'$  are positive constants ( $k^2 + k'^2 = 1$ ) given by

$$\alpha^2 = \frac{4\varphi_0 + 2\varphi_1 - 3\omega}{12}, \quad k^2 = \frac{2\varphi_0 - 2\varphi_1}{4\varphi_0 + 2\varphi_1 - 3\omega}, \quad k'^2 = \frac{2\varphi_0 + 4\varphi_1 - 3\omega}{4\varphi_0 + 2\varphi_1 - 3\omega}.$$

Therefore

$$(41) \quad |\varphi(x)| = \varphi_1 + (\varphi_0 - \varphi_1)cn^2(\alpha x; k).$$

Consider in  $[0, T] = [0, 2K(k)/\alpha]$  the differential operators introduced earlier

$$(42) \quad \mathcal{L}_- = -\frac{d^2}{dx^2} + (\omega - 2|\varphi|), \quad \mathcal{L}_+ = -\frac{d^2}{dx^2} + (\omega - |\varphi|),$$



supplied with periodic boundary conditions. By the above formulas,  $\varphi_0 - \varphi_1 = 6\alpha^2 k^2$ ,  $2\varphi_0 - \omega = 4\alpha^2(1 + k^2)$ . Taking  $y = \alpha x$  as an independent variable in  $\mathcal{L}_-$ , one obtains  $\mathcal{L}_- = \alpha^2 \Lambda_1$  with an operator  $\Lambda_1$  in  $[0, 2K(k)]$  given by

$$\begin{aligned}\Lambda_1 &= -\frac{d^2}{dy^2} + \alpha^{-2}[\omega - 2(\varphi_1 + (\varphi_0 - \varphi_1)cn^2(y; k))] \\ &= -\frac{d^2}{dy^2} + \frac{\omega - 2\varphi_0}{\alpha^2} + \frac{2(\varphi_0 - \varphi_1)}{\alpha^2}sn^2(y; k) \\ &= -\frac{d^2}{dy^2} - 4(1 + k^2) + 12k^2sn^2(y; k).\end{aligned}$$

The spectral properties of the operator  $\Lambda_1$  in  $[0, 2K(k)]$  are well-known. The first three eigenvalues are simple and moreover the corresponding eigenfunctions of  $\Lambda_1$  are given by

$$\begin{aligned}\mu_0 &= \kappa^2 - 2 - 2\sqrt{1 - \kappa^2 + 4\kappa^4} < 0 \\ \psi_0(y) &= dn(y; \kappa)[1 - (1 + 2\kappa^2 - \sqrt{1 - \kappa^2 + 4\kappa^4})sn^2(y; \kappa)] > 0 \\ \mu_1 &= 0 \\ \psi_1(y) &= dn(y; \kappa)sn(y; \kappa)cn(y; \kappa) = \frac{1}{2}\frac{d}{dy}cn^2(y; \kappa) \\ \mu_2 &= \kappa^2 - 2 + 2\sqrt{1 - \kappa^2 + 4\kappa^4} > 0 \\ \psi_2(y) &= dn(y; \kappa)[1 - (1 + 2\kappa^2 + \sqrt{1 - \kappa^2 + 4\kappa^4})sn^2(y; \kappa)].\end{aligned}$$

Since the eigenvalues of  $\mathcal{L}_-$  and  $\Lambda_1$  are related via  $\lambda_n = \alpha^2 \mu_n$ , it follows that the first three eigenvalues of the operator  $\mathcal{L}_-$ , equipped with periodic boundary condition on  $[0, 2K(k)]$  are simple and  $\lambda_0 < 0$ ,  $\lambda_1 = 0$ ,  $\lambda_2 > 0$ . The corresponding eigenfunctions are  $\psi_0(\alpha x)$ ,  $\psi_1(\alpha x) = C\varphi'$  and  $\psi_2(\alpha x)$ . In a similar way, since  $\mathcal{L}_+ = \alpha^2 \Lambda_2$ , one obtains that in  $[0, 2K(k)]$

$$\Lambda_2 = -\frac{d^2}{dy^2} - 2(1 + k^2) + 6k^2sn^2(y; k) + \omega/2\alpha^2.$$

To express  $\omega$  through  $\alpha$  and  $k$ , one should take into account the fact that in the cubic equation we used to determine  $\varphi_0$  and  $\varphi_1$ , we have that the coefficient at  $\rho$  is zero. Therefore,

$$\varphi_0\varphi_1 + (\varphi_0 + \varphi_1)(\frac{3}{2}\omega - \varphi_0 - \varphi_1) = 0.$$

As  $\varphi_0 = 2\alpha^2 + 2\alpha^2 k^2 + \frac{1}{2}\omega$ ,  $\varphi_1 = 2\alpha^2 - 4\alpha^2 k^2 + \frac{1}{2}\omega$ , after replacing these values in the above equation one obtains  $\omega^2 = 16\alpha^4(1 - k^2 + k^4)$ . Since  $\omega > 0$ , we finally obtain

$$\Lambda_2 = -\frac{d^2}{dy^2} + 2(-1 - k^2 + \sqrt{1 - k^2 + k^4}) + 6k^2sn^2(y; k).$$

On the other hand, (41) yields

$$|\varphi| = 2\alpha^2[1 + k^2 + \sqrt{1 - k^2 + k^4} - 3k^2sn^2(y; k)].$$

The first three eigenvalues and corresponding eigenfunctions of  $\Lambda_2$  are as follows:

$$\begin{aligned}\lambda_0 &= 0, & \psi_0 &= \varphi, \\ \lambda_1 &= 2 - k^2 + 2\sqrt{1 - k^2 + k^4}, & \psi_1 &= dn'(y; k) \\ \lambda_2 &= 4\sqrt{1 - k^2 + k^4}, & \psi_2 &= 1 + k^2 - \sqrt{1 - k^2 + k^4} - 3k^2sn^2(y; k).\end{aligned}$$

The considerations above yield

**Proposition 4.** *The linear operator  $\mathcal{L}_-$  defined by (42) has the following spectral properties:*

- (i) *The first three eigenvalues of  $\mathcal{L}_-$  are simple.*
- (ii) *The second eigenvalue of  $\mathcal{L}_-$  is  $\lambda_1 = 0$ .*
- (iii) *The rest of the spectrum of  $\mathcal{L}_-$  consists of a discrete set of positive eigenvalues.*

*The linear operator  $\mathcal{L}_+$  defined by (42) has the following spectral properties:*

- (i)  *$\mathcal{L}_+$  has no negative eigenvalue.*
- (ii) *The first eigenvalue of  $\mathcal{L}_+$  is zero, which is simple.*
- (iii) *The rest of the spectrum of  $\mathcal{L}_+$  consists of a discrete set of positive eigenvalues.*

4.2.2. *Cubic Schrödinger equation.* Consider the cubic nonlinear Schrödinger equation

$$(43) \quad iu_t + u_{xx} + |u|^2 u = 0,$$

where  $u = u(x, t)$  is a complex-valued function of  $(x, t) \in \mathbb{R}^2$ .

For  $\varphi$  one obtains the equation

$$(44) \quad \varphi'' - \omega\varphi + \varphi^3 = 0.$$

Integrating once again, we obtain

$$(45) \quad \varphi'^2 - \omega\varphi^2 + \frac{1}{2}\varphi^4 = c$$

and  $\varphi$  is a periodic function provided that the energy level set  $H(x, y) = c$  of the Hamiltonian system  $dH = 0$ ,

$$H(x, y) = y^2 - \omega x^2 + \frac{1}{2}x^4,$$

contains an oval (a simple closed real curve free of critical points). The level set  $H(x, y) = c$  contains two periodic trajectories if  $\omega > 0$ ,  $c \in (-\frac{1}{2}\omega^2, 0)$  and a unique periodic trajectory if  $\omega \in \mathbb{R}$ ,  $c > 0$ . Under these conditions, the solution of (44) is determined by  $H(\varphi, \varphi') = c$  and  $r$  is periodic of period  $T = T(\omega, c)$ .

Below, we are going to consider the case  $c < 0$ . Let us denote by  $\varphi_0 > \varphi_1 > 0$  the positive roots of  $\frac{1}{2}\varphi^4 - \omega\varphi^2 - c = 0$ . Then, up to a translation, we obtain the respective explicit formulas

$$(46) \quad \varphi(z) = \mp \varphi_0 dn(\alpha z; k), \quad k^2 = \frac{\varphi_0^2 - \varphi_1^2}{\varphi_0^2} = \frac{-2\omega + 2\varphi_0^2}{\varphi_0^2}, \quad \alpha = \frac{\varphi_0}{\sqrt{2}}, \quad T = \frac{2K(k)}{\alpha}.$$

Here and below  $K(k)$  and  $E(k)$  are, as usual, the complete elliptic integrals of the first and second kind in a Legendre form. By (46), one also obtains  $\omega = (2 - k^2)\alpha^2$  and, finally,

$$(47) \quad T = \frac{2\sqrt{2 - k^2}K(k)}{\sqrt{\omega}}, \quad k \in (0, 1), \quad T \in I = \left( \frac{2\pi}{\sqrt{\omega}}, \infty \right).$$

Again,  $\mathcal{L}_-$  and  $\mathcal{L}_+$  are given by

$$(48) \quad \begin{aligned} \mathcal{L}_- &= -\partial_x^2 + (\omega - 3\varphi^2), \\ \mathcal{L}_+ &= -\partial_x^2 + (\omega - \varphi^2), \end{aligned}$$

with periodic boundary conditions in  $[0, T]$ .

We use now (46) and (47) to rewrite operators  $\mathcal{L}_\pm$  in more appropriate form. From the expression for  $\varphi(x)$  from (46) and the relations between elliptic functions  $sn(x)$ ,  $cn(x)$  and  $dn(x)$ , we obtain

$$\mathcal{L}_- = \alpha^2[-\partial_y^2 + 6k^2 sn^2(y) - 4 - k^2]$$

where  $y = \alpha x$ .

It is well-known that the first five eigenvalues of  $\Lambda_1 = -\partial_y^2 + 6k^2 sn^2(y, k)$ , with periodic boundary conditions on  $[0, 4K(k)]$ , where  $K(k)$  is the complete elliptic integral of the first kind, are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{aligned} \nu_0 &= 2 + 2k^2 - 2\sqrt{1 - k^2 + k^4}, & \phi_0(y) &= 1 - (1 + k^2 - \sqrt{1 - k^2 + k^4})sn^2(y, k), \\ \nu_1 &= 1 + k^2, & \phi_1(y) &= cn(y, k)dn(y, k) = sn'(y, k), \\ \nu_2 &= 1 + 4k^2, & \phi_2(y) &= sn(y, k)dn(y, k) = -cn'(y, k), \\ \nu_3 &= 4 + k^2, & \phi_3(y) &= sn(y, k)cn(y, k) = -k^{-2}dn'(y, k), \\ \nu_4 &= 2 + 2k^2 + 2\sqrt{1 - k^2 + k^4}, & \phi_4(y) &= 1 - (1 + k^2 + \sqrt{1 - k^2 + k^4})sn^2(y, k). \end{aligned}$$

It follows that the first three eigenvalues of the operator  $L_-$ , equipped with periodic boundary condition on  $[0, 2K(k)]$  (that is, in the case of left and right family), are simple and  $\lambda_0 = \alpha^2(\nu_0 - \nu_3) < 0$ ,  $\lambda_1 = \alpha^2(\nu_3 - \nu_3) = 0$ ,  $\lambda_2 = \alpha^2(\nu_4 - \nu_3) > 0$ . The corresponding eigenfunctions are  $\psi_0 = \phi_0(\alpha x)$ ,  $\psi_1 = \varphi'(x)$ ,  $\psi_2 = \phi_4(\alpha x)$ .

Similarly, for the operator  $\mathcal{L}_+$  we have

$$\mathcal{L}_+ = \alpha^2[-\partial_y^2 + 2k^2 sn^2(y, k) - k^2]$$

in the case of left and right family. The spectrum of  $\Lambda_2 = -\partial_y^2 + 2k^2 sn^2(y, k)$  is formed by bands  $[k^2, 1] \cup [1 + k^2, +\infty)$ . The first three eigenvalues and the corresponding eigenfunctions with periodic boundary conditions on  $[0, 4K(k)]$  are simple and

$$\begin{aligned} \epsilon_0 &= k^2, & \theta_0(y) &= dn(y, k), \\ \epsilon_1 &= 1, & \theta_1(y) &= cn(y, k), \\ \epsilon_2 &= 1 + k^2, & \theta_2(y) &= sn(y, k). \end{aligned}$$

From (45) it follows that zero is an eigenvalue of  $\mathcal{L}_+$  and it is the first eigenvalue in the case of left and right family, with corresponding eigenfunction  $\varphi(x)$ .

The above considerations gives an identical result to Proposition 4 for the operators  $\mathcal{L}_\pm$  defined in (48). Thus, in both the quadratic and cubic cases, we have obtained that there is a single

negative eigenvalue for the matrix operator  $\begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix}$  and thus our proof is complete.

## 5. THE DEFOCUSING MODIFIED KDV EQUATION

Now consider the defocusing modified Korteweg-de Vries equation

$$(49) \quad u_t - 3u^2 u_x + u_{xxx} = 0.$$

We are looking for traveling wave solutions of the form  $u(x, t) = \phi(x - ct)$ ,  $c < 0$ .

So, if we substitute this specific solution in the defocusing mKdV and consider the integration constant equal to zero then  $Q = Q_c$  satisfies the ordinary differential equation

$$(50) \quad \phi'' - c\phi - \phi^3 = 0.$$

From this we obtain the first order differential equation (in the associated quadrature form)

$$(51) \quad [\phi']^2 = \frac{1}{2}(\phi^4 + 2c\phi^2 + A),$$

where  $A$  is the integration constant and which need to be different of zero for obtaining periodic profile solutions. Analogously as in the case of modified Korteweg-de Vries equation, we obtain the explicit form for the periodic traveling wave solutions

$$\phi_c(\xi) = \eta_2 \operatorname{sn}(\alpha\xi; k),$$

where  $\eta_1 > \eta_2 > 0$  are positive roots of the polynomial  $F(t) = t^4 + 2ct^2 + A$  and  $\alpha = \frac{\eta_1}{\sqrt{2}}$ ,  $k^2 = \eta_2^2/\eta_1^2 \in (0, 1)$ . Since the function  $\operatorname{sn}(x)$  has minimal period  $4K(k)$  then the minimal period of  $\phi$ ,  $T$ , is given by  $T = 4K(k)/\alpha$ . Moreover,

$$k^2 = \frac{-2c - \eta_1^2}{\eta_1^2}$$

Now consider the spectral problem for the operator  $\mathcal{L} = -\partial_x^2 + 3\phi^2 + c$ ,

**Proposition 5.** *Let  $\phi$  be the snoidal wave solution of the defocusing Korteweg-de Vries equation. Let*

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots,$$

*connote the eigenvalues of the operator  $\mathcal{L}$ . Then*

$$\lambda_0 < \lambda_1 = 0 < \lambda_2 < \lambda_3 < \lambda_4$$

*are all simple whilst, for  $j \geq 5$ , the  $\lambda_j$  are double eigenvalues. The  $\lambda_j$  only accumulate at  $+\infty$ .*

*Proof.* Since  $\mathcal{L} \frac{d}{dx} \phi = 0$  and  $\frac{d}{dx} \phi$  has 2 zeros in  $[0, T)$ , it follows that 0 is either  $\lambda_1$  or  $\lambda_2$ . We will show that  $0 = \lambda_1 < \lambda_2$ .

From the expression for  $\phi(x)$ , we obtain

$$\mathcal{L} = \alpha^2 [-\partial_y^2 + 6k^2 \operatorname{sn}^2(y) - 1 - k^2]$$

where  $y = \alpha x$ .

The eigenvalues and corresponding eigenfunctions are:

$$\begin{aligned} \nu_0 &= 2 + 2k^2 - 2\sqrt{1 - k^2 + k^4}, & \psi_0(y) &= 1 - (1 + k^2 - \sqrt{1 - k^2 + k^4}) \operatorname{sn}^2(y, k), \\ \nu_1 &= 1 + k^2, & \psi_1(y) &= cn(y, k) dn(y, k) = \operatorname{sn}'(y, k), \\ \nu_2 &= 1 + 4k^2, & \psi_2(y) &= \operatorname{sn}(y, k) dn(y, k) = -cn'(y, k), \\ \nu_3 &= 4 + k^2, & \psi_3(y) &= \operatorname{sn}(y, k) cn(y, k) = -k^{-2} dn'(y, k), \\ \nu_4 &= 2 + 2k^2 + 2\sqrt{1 - k^2 + k^4}, & \psi_4(y) &= 1 - (1 + k^2 + \sqrt{1 - k^2 + k^4}) \operatorname{sn}^2(y, k). \end{aligned}$$

It follows that the first five eigenvalues of the operator  $\mathcal{L}$ , equipped with periodic boundary condition on  $[0, 4K(k)]$  are simple and  $\lambda_0 = \alpha^2(\nu_0 - \nu_1) < 0$ ,  $\lambda_1 = \alpha^2(\nu_1 - \nu_1) = 0$ ,  $\lambda_2 = \alpha^2(\nu_2 - \nu_1) > 0$ ,  $\lambda_3 = \alpha^2(\nu_3 - \nu_1) > 0$ ,  $\lambda_4 = \alpha^2(\nu_4 - \nu_1) > 0$ . The corresponding eigenfunctions are  $\chi_0 = \psi_0(\alpha x)$ ,  $\chi_1 = \phi'(x)$ ,  $\chi_2 = \psi_2(\alpha x)$ ,  $\chi_3 = \psi_3(\alpha x)$ ,  $\chi_4 = \psi_4(\alpha x)$ .

In the case of Defocusing Modified Korteweg-de Vries equation inequality (37) is equivalent to the inequality

$$\begin{aligned} & \frac{|(\sqrt{1 - \kappa^2 + \kappa^4} - 1)K(\kappa) + (1 + \kappa^2 - \sqrt{1 - \kappa^2 + \kappa^4})E(\kappa)|}{\sqrt{|1 + \kappa^2 - 2\sqrt{1 - \kappa^2 + \kappa^4}|}} < \\ & < \frac{|(\sqrt{1 - \kappa^2 + \kappa^4} + 1 + \kappa^2)E(\kappa) - (1 + \sqrt{1 - \kappa^2 + \kappa^4})K(\kappa)|}{\sqrt{1 + \kappa^2 + 2\sqrt{1 - \kappa^2 + \kappa^4}}} \end{aligned}$$

However, one can see from the picture below, that this inequality does not hold for any value of  $\kappa$ . Thus, our method fails to conclude the transversal instability of such waves.  $\square$

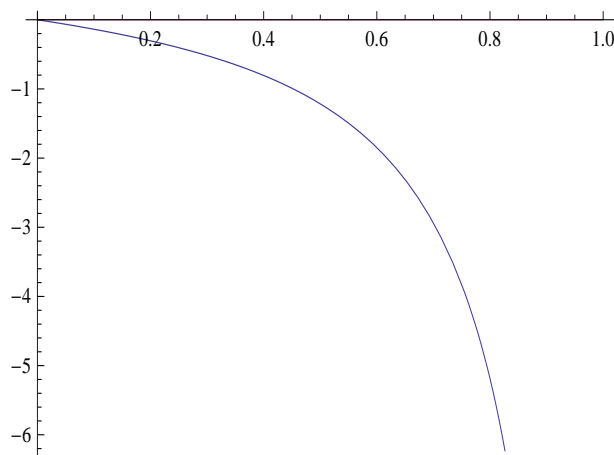


FIGURE 3. Here, a plot of the difference of the two quantities is given. A positive function implies instability.

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